A FAMILY OF COMPLEX WAVELETS FOR THE CHARACTERIZATION OF SINGULARITIES

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Generally, the irregular behaviour of a function $f(x)$ is described by its Hölder exponent $h(x)$ quantifying the strength of the singularity at a point x. In most cases, where the function does not contain oscillating singularities, the Wavelet Transform Modulus Maxima method (WTMM) allows a reliable estimation[1,2] of the singularity spectrum $D(h)$, i.e. the Hausdorff dimension of the set of all points x with the same Hölder exponent $h(x) = h$. Methods based on the WTMM should be numerically more stable than direct methods such as the Structure Function method[1], since they involve only weighed-averaged quantities and not averages of function increments and since small errors are relatively less important if the calculation is restricted to maxima.

However, in the presence of oscillating singularities the standard WTMM method gives irrelevant information on the Hölder regularity of the function. In general, *two* exponents h, β are necessary to describe the singular behaviour of a function $f(x)$, namely the Hölder exponent h and the oscillation exponent β describing the local power law divergence of the instantaneous frequency[3,4]. If $f(x)$ contains oscillating singularities the regularity of the primitive of $f(x)$ depends on β. In this case the Hölder exponent does not increase by 1 as in the case of a cusp singularity but by $\beta + 1$. Thus, the singularity spectrum of general functions depends on both exponents, $D(h, \beta)$.

In order to extract Hölder exponents and to quantify at the same time the oscillating behaviour we propose to use a family of *complex* progressive wavelets[5] $\{\psi_n\}$

$$\psi_0(x) = (e^{i\omega_0 x} - e^{-\omega_0^2/2})e^{-x^2/2}, \qquad \psi_n(x) = \frac{d}{dx}\psi_{n-1}(x) \qquad (n \in \mathbf{Z}) \tag{1}$$

with an increasing number of vanishing moments

$$\int_{-\infty}^{+\infty} x^k \psi_n(x) dx = 0 \qquad \text{for} \quad 0 \leq k \leq n \tag{2}$$

For a sufficiently large frequency ω_0 ($\omega_0 > 5$ for practical purposes), ψ_0 can be replaced by the Morlet wavelet.

Using these wavelets we follow similar ideas as in an earlier paper[2], where an algorithm was developed for a family of real valued wavelets based on derivatives of the Gaussian distribution, which allows a *direct tracing* of the skeleton of wavelet transform maxima lines[2]. Using the following abbreviation for the continuous wavelet transform

$$W^n f := W^n f(a,b) = \frac{1}{a} \int_{-\infty}^{+\infty} f(x) \overline{\psi_n(\frac{x-b}{a})} dx \qquad (a, b \in \mathbf{R}, a > 0) \tag{3}$$

where a denotes the scale, b the shift parameter, and $\overline{\psi_n}$ the complex conjugate of ψ_n, we set up a partial differential equation for the wavelet transform

$$\left(a \frac{\partial^2}{\partial b^2} - \frac{\partial}{\partial a} - i\omega_0 \frac{\partial}{\partial b} + \frac{n}{a} \right) W^n f = 0 \tag{4}$$

Describing the maxima line in a parametric form $\{a(t), b(t)\}$, the motion along this line is given by two ordinary differential equations

$$\frac{da}{dt} = -C \frac{\partial}{\partial b} \left(\mathrm{Real}(\frac{\partial W^n f}{\partial b} \overline{W^n f}) \right) \quad , \quad \frac{db}{dt} = C \frac{\partial}{\partial a} \left(\mathrm{Real}(\frac{\partial W^n f}{\partial b} \overline{W^n f}) \right) \tag{5}$$

with a constant C, which can be integrated numerically. Similar equations can be derived for the ridges [5]. The skeleton of maxima lines can be continuously traced up to the desired accuracy with a reduced computational effort. Based on these maxima lines a partition function $Z(p, q, a)$ is defined

$$Z(p,q,a) = \sum_{i \in \mathrm{max.lines}} (\sup_{a' < a} |W^n f(a, b_i)|)^q d_i(a)^p \tag{6}$$

where $d_i(a)$ denotes the distance between the i-th and the $(i+1)$-th maxima line. In comparison with *classical* methods, the main advantage of the proposed direct tracing algorithm is, that the supremum along each maxima line can be easily evaluated.

Exploiting the scaling behaviour of the partition function $Z(p,q,a) \sim a^{\tau(p,q)}$ and applying the Legendre transform to the exponent $\tau(p,q)$, one obtains the two-dimensional $D(h,\beta)$ singularity spectrum[3]

$$D(h,\beta) = \min_{p,q}(qh + p(\beta+1) - \tau(p,q)) \tag{7}$$

In practice, due to the finite length and noise in data sets, the accuracy of the Legendre transform might decrease. In order to avoid this problem, we propose a generalization of the canonical method of Chhabra and Jensen[6] to two-dimensional spectra which allows a direct computation of the singularity spectrum $D(h,\beta)$.

References

1. J.F. Muzy, E. Bacry and A. Arneodo, *Int.J.Bif. and Chaos* **4**, 245 (1994).
2. M. Haase, B. Lehle, in: *Fractals and Beyond*, (M. M. Novak Ed.), World Scientific, Singapore 241 (1998).
3. A. Arneodo, E. Bacry, S. Jaffard, and J.F. Muzy, *J. Statist. Phys.* **87**, 179 (1997).
4. A. Arneodo, E. Bacry and J.F. Muzy, *Phys.Rev.Lett.* **74**, 4823 (1995).
5. B. Torrésani, in: *Progress in Wavelet Analysis and Applications*, (Y. Meyer and S. Roques Eds.) Editions Frontières (1993).
6. A. Chhabra and J.V. Jensen: *Phys.Rev.Lett.* **62**, 1327 (1989).